Pergamon

Computers Math. Applic. Vol. 33, No. 1/2, pp. 165–180, 1997

0898-1221/97 $17.00 + 0.00

PII: S0898-1221(96)00228-3

Some Results in Segmented Approximation

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Abstract—Univariate segmented approximations are discussed in some generality in order to present a review, respectively, survey on the main ideas of this topic. The basic results are given with complete proofs. The description of a number of instructive examples is followed by new *a priori* error estimates and by asymptotic results concerning the minimal deviation, provided the approximating functions belong to a Pólya space.

Keywords—Segmented approximation.

1. INTRODUCTION

For a few years, the problem of segmented approximation has gained some new interest. Simple piecewise continuous functions are still a convenient tool to solve many approximation problems. Today the focus of investigations is on constructive methods and on error estimations. One wants to know, how much on accuracy can be gained in using segmentation, and how to get these approaches concretely.

In this paper, we are restricted to the univariate case and uniform approximation. We will give—to some extent in detail—a survey on the main ideas of this field, present a few instructive examples, and prove some new theorems concerning *a priori* estimations of the approximation error.

Let us be given a compact interval $J = [a, b] \subset \mathbb{R}$. For a fixed $k \in \mathbb{N}$, $k \geq 2$ we consider $k-1$ real numbers $\tau_1, \tau_2, \dots, \tau_{k-1}$, called knots. The set of such knots

$$T_k = \{\tau_1, \tau_2, \dots, \tau_{k-1}\}$$

is referred to as a *partition* of J, provided

$$a = \tau_0 \leq \tau_1 \leq \dots \leq \tau_{k-1} \leq \tau_k = b$$

holds. If we consider the unpartitioned case, we sometimes permit k to have the value 1.

For any subset $B \subset J$, we will make use of the Chebyshev norm

$$\|f\|_B = \sup_{x \in B} |f(x)|$$

for $f \in C(J)$. On every subinterval $J_\nu = [\tau_{\nu-1}, \tau_\nu]$, $\nu = 1, 2, \dots, k$, we are given subsets $V_\nu \subset C(U(J_\nu))$, provided $\tau_\nu > \tau_{\nu-1}$. Here $U(J_\nu)$ denotes a neighbourhood of J_ν with $U(J_\nu) \subset J$.

We want to approximate a given function $f \in C(J)$ by functions of the set V_ν on the subinterval J_ν, $\nu = 1, 2, \dots, k$. The minimal deviation in approximating the function f on the interval J_ν is defined by

$$\rho_\nu(f) = \rho(f; \tau_{\nu-1}, \tau_\nu) = \inf_{v \in V_\nu} \|f - v\|_{J_\nu}$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX for all ν with $\tau_\nu > \tau_{\nu-1}$. We put $\rho_\nu(f) = 0$, if $\tau_\nu = \tau_{\nu-1}$ holds. We need an additional assumption on the sets V_ν: for every real number ω, the continuous function $g : J_\nu \to \mathbb{R}$, defined by

$$g(x) = \omega, \qquad \text{for all } x \in J_\nu,$$

has the property that

$$\lim_{\tau_\nu \to \tau_{\nu-1}} \rho(g; \tau_{\nu-1}, \tau_\nu) = 0.$$

The problem of *segmented approximation* consists of finding a partition

$$T_k^* = \left\{\tau_1^*, \tau_2^*, \ldots, \tau_{k-1}^*\right\}$$

such that the quantity

$$\max_{\nu=1,2,\ldots,k} \rho_\nu(f)$$

for this partition is as small as possible. Let us define the minimal deviation of the segmented approximation problem by

$$\rho^*(f) = \inf_{T_k} \max_{\nu=1,2,\ldots,k} \rho_\nu(f).$$

The problem was stated and investigated first by Lawson [1] in 1964. He considered segmented rational approximation problems; i.e., in every subinterval one may use another family of rational functions for approximation. A general approach was given by McLure [2], mostly for L_2-approximation and by Meinardus [3] in the uniform norm. In the latter, one could use almost arbitrary nonlinear families of continuous functions. The motivation for a segmented approximation is of course that one hopes to gain a significant amount of accuracy, compared with the unpartitioned case.

2. THE ELEMENTARY THEORY

We consider the simplex

$$M = \left\{(\xi, \eta) \in \mathbb{R}^2 \mid a \le \xi \le \eta \le b\right\}.$$

THEOREM 1. *[1–3] For fixed $f \in C(J)$, the following assertions hold true:*

(1) *$\rho(f; \xi, \eta)$ is a continuous and nonnegative function on M;*
(2) *$\rho(f; \xi, \xi) = 0$, for all $\xi \in [a, b]$;*
(3) *$\rho(f; \xi, \eta) \le \rho(f; \tilde{\xi}, \tilde{\eta})$ whenever $a \le \tilde{\xi} \le \xi \le \eta \le \tilde{\eta} \le b$ holds true.*

Here the numbers ξ and η serve as abbreviations for $\tau_{\nu-1}$ and τ_ν, respectively. The numbers ξ' and η' are assumed to belong to the neighbourhood $U(J_\nu)$.

PROOF. The assertions (2) and (3) are trivial. Put, for $(\xi, \eta) \in M$ with $\xi < \eta$ and $(\xi', \eta') \in M$, where ξ' and η' belong to $U(J_\nu)$,

$$[\xi, \eta] \cup [\xi', \eta'] = [u, w]$$

and

$$[\xi, \eta] \cap [\xi', \eta'] = [u', w'].$$

Then, from assertion (3), we get the inequalities

$$\rho(f; u', w') \le \rho(f; \xi, \eta) \le \rho(f; u, w)$$

and

$$\rho(f; u', w') \le \rho(f; \xi', \eta') \le \rho(f; u, w),$$

so that

$$|\rho(f;\xi,\eta) - \rho(f;\xi',\eta')| \leq \rho(f;u,w) - \rho(f;u',w') \tag{1}$$

holds. To a prescribed $\varepsilon > 0$, there is a $v \in V_\nu$ such that

$$\|f - v\|_{[u',w']} \leq \rho(f;u',w') + \frac{\varepsilon}{3}.$$

Let $x \in (w', w]$. Then, to the same ε, there is a number $\delta_1 > 0$ such that, due to continuity,

$$|f(x) - f(w')| < \frac{\varepsilon}{3} \quad \text{and} \quad |v(x) - v(w')| < \frac{\varepsilon}{3},$$

if only

$$|w - w'| < \delta_1.$$

Thus, for $x \in (w', w]$, we get

$$|f(x) - v(x)| \leq |f(x) - f(w')| + |v(x) - v(w')| + |f(w') - v(w')| < \rho(f;u',w') + \varepsilon.$$

Now let $x \in [u, u')$. Then to the same ε there is a number $\delta_2 > 0$ such that, due to continuity,

$$|f(x) - f(u')| < \frac{\varepsilon}{3} \quad \text{and} \quad |v(x) - v(u')| < \frac{\varepsilon}{3},$$

if only

$$|u - u'| < \delta_2.$$

Thus, for $x \in [u, u')$, we also get

$$|f(x) - v(x)| \leq |f(x) - f(u')| + |v(x) - v(u')| + |f(u') - v(u')| < \rho(f;u',w') + \varepsilon.$$

Hence,

$$\rho(f;u,w) \leq \|f - v\|_{[u,w]} < \rho(f;u',w') + \varepsilon,$$

which, in using (1), leads to

$$|\rho(f;\xi,\eta) - \rho(f;\xi',\eta')| < \varepsilon,$$

if only

$$|\xi - \xi'| < \delta_2 \quad \text{and} \quad |\eta - \eta'| < \delta_1$$

holds. We conclude that $\rho(f;\xi,\eta)$ is continuous on M. Obviously this function is always nonnegative. ∎

THEOREM 2. *[1–3] To every $f \in C(J)$ there exists a partition T_k^* of J such that*

$$\max_{\nu=1,2,\ldots,k} \rho\left(f;\tau_{\nu-1}^*,\tau_\nu^*\right) = \rho^*(f). \tag{2}$$

PROOF. Since the expression

$$\max_{\nu=1,2,\ldots,k} \rho\left(f;\tau_{\nu-1},\tau_\nu\right)$$

represents a continuous and nonnegative function on the set

$$S = \left\{(\tau_1, \tau_2, \ldots, \tau_{k-1}) \in \mathbb{R}^{k-1} \mid a \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{k-1} \leq b\right\},$$

the infimum is attained at some partition T_k^*. ∎

A partition which satisfies (2) is called *optimal*.

THEOREM 3. *[1–3] For any fixed partition T_k of J, the following inclusion holds:*

$$\min_{\nu=1,2,\dots,k} \rho(f;\tau_{\nu-1},\tau_\nu) \le \rho^*(f) \le \max_{\nu=1,2,\dots,k} \rho(f;\tau_{\nu-1},\tau_\nu). \tag{3}$$

PROOF. We only need to prove the first inequality. Let T_k be any partition and T_k^* be an optimal partition. Then there is a subscript $\mu \in \{1,2,\dots,k\}$ such that

$$\tau_{\mu-1}^* \le \tau_{\mu-1} \quad \text{and} \quad \tau_\mu \le \tau_\mu^*. \tag{4}$$

Using such subscript, we get

$$\rho^*(f) \ge \rho\left(f;\tau_{\mu-1}^*,\tau_\mu^*\right) \ge \rho(f;\tau_{\mu-1},\tau_\mu) \ge \min_{\nu=1,2,\dots,k} \rho(f;\tau_{\nu-1},\tau_\nu),$$

which proves Theorem 3. ∎

LEMMA 1. *We consider the interval $[a,\tau]$ for $\tau \in (a,b]$. Let $f \in C(J)$ and denote by $\rho_k^*(f;\tau)$ the minimal deviation of the segmental approximation of f with respect to the interval $[a,\tau]$. Define $\rho_k^*(f;a) = 0$. Then $\rho_k^*(f;\tau)$ depends continuously on τ for $\tau \in [a,b]$.*

PROOF. Let τ and τ' be two different points in $(a,b]$. Without restriction we may assume $\tau' > \tau$. Now let τ_ν, $\nu = 1,2,\dots,k-1$, with

$$a = \tau_0 \le \tau_1 \le \cdots \le \tau_{k-1} \le \tau_k = \tau$$

form an optimal partition of the interval $[a,\tau]$. According to Theorem 2, such partition exists. Due to the definition there is a subscript $\mu \in \{1,2,\dots,k\}$ such that

$$\rho_k^*(f;\tau) = \rho\left(f;\tau_{\mu-1},\tau_\mu\right). \tag{5}$$

We consider the same partition also for the larger interval $[a,\tau']$:

$$a = \tau_0 \le \tau_1 \le \cdots \le \tau_{k-1} \le \tau_k' = \tau'.$$

Then, if

$$\rho\left(f;\tau_{k-1},\tau_k\right) < \rho\left(f;\tau_{\mu-1},\tau_\mu\right),$$

there is a number $\delta > 0$ such that we have

$$\rho_k^*\left(f;\tau'\right) = \rho\left(f;\tau_{\mu-1},\tau_\mu\right) = \rho_k^*(f;\tau)$$

holds, provided $\tau' - \tau < \delta$.

But if we may choose $\mu = k$, Theorem 3 yields

$$\rho_k^*\left(f;\tau'\right) \le \rho\left(f;\tau_{k-1},\tau'\right).$$

This leads to the inequalities

$$0 \le \rho_k^*\left(f;\tau'\right) - \rho_k^*(f;\tau) \le \rho\left(f;\tau_{k-1},\tau'\right) - \rho\left(f;\tau_{k-1},\tau\right).$$

Due to the continuity of the minimal deviation $\rho(f;\tau_{k-1},\tau)$, proved in Theorem 1, we see that $\rho_k^*(f;\tau)$ is continuous too. The continuity at $\tau = 0$ is obvious. ∎

A partition T_k of (J) is called *leveled*, if the $k-1$ conditions

$$\rho(f;\tau_{\nu-1},\tau_\nu) = \rho(f;\tau_\nu,\tau_{\nu+1})$$

are satisfied for $\nu = 1,2,\dots,k-1$.

THEOREM 4. *[1–3] There exists, for any $f \in C(J)$, a leveled partition T_k^* of J with*

$$a = \tau_0^* < \tau_1^* < \tau_2^* < \cdots < \tau_{k-1}^* < \tau_k^* = b. \tag{6}$$

PROOF. We first prove by induction that there is a leveled partition T_k^* of J satisfying the weaker inequalities

$$a = \tau_0^* \le \tau_1^* \le \tau_2^* \le \cdots \le \tau_{k-1}^* \le \tau_k^* = b.$$

Consider the continuous function of the variable τ,

$$u(\tau) := \rho(f; a, \tau) - \rho(f; \tau, b)$$

on the interval $[a, b]$. Since $u(a) \le 0$ and $u(b) \ge 0$, it is obvious that there exists a number $\tau = \tau_1^* \in [a, b]$ with $u(\tau_1^*) = 0$.

Next consider the continuous function of the variable τ,

$$w(\tau) := \rho_k^*(f; \tau) - \rho(f; \tau, b)$$

on the interval $[a, b]$. Since $w(a) \le 0$ and $w(b) \ge 0$, it is obvious that there exists a number $\tau = \tau_k^* \in [a, b]$ with $w(\tau_k^*) = 0$. If $\tau_k^* = a$, we define $\tau_\nu^* = a$ for $\nu = 0, 1, \ldots, k$, and $\tau_{k+1}^* = b$. These knots, having in mind that the number k of the assertion has to be replaced by $k+1$, form a leveled partition of the interval J. If, however, $\tau_k^* > a$, then we choose, using the induction hypothesis, a leveled partition of the interval $[a, \tau]$, again denoted by τ_ν^*, $\nu = 1, 2, \ldots, k-1$. Then the knots $\tau_1^*, \tau_2^*, \ldots, \tau_k^*$ form a leveled partition of J,

$$a = \tau_0^* \le \tau_1^* \le \cdots \le \tau_{k-1}^* \le \tau_k^* \le \tau_{k+1}^* = b.$$

The induction is now complete.

Let $m \le k$ be the number of those subintervals $J_\nu \subset J$ which have positive length. Then, for $m = k$, the assertion of Theorem 4 is proved.

If $m < k$, then some of the subintervals have collapsed to a zero length. In this case, $\rho(f; \tau_{\nu-1}^*, \tau_\nu^*) = 0$ for $\nu = 1, 2, \ldots, k$, however. Of course there is at least one subinterval of positive length left. Now choose $m - k$ arbitrary, but mutually different, points within the interior of such subinterval. Then renumber all the endpoint of the new k subintervals. The results yield new knots $\tilde{\tau}_\nu^*$, $\nu = 0, 1, \ldots, k$, which satisfy (6). Furthermore, all the minimal deviations $\rho(f; \tilde{\tau}_{\nu-1}^*, \tilde{\tau}_\nu^*)$, $\nu = 1, 2, \ldots, k$, vanish. ∎

THEOREM 5. *[1–3] Every leveled partition is optimal.*

PROOF. The assertion is a simple consequence of Theorem 3. ∎

3. SOME EXAMPLES

We present a few examples in which the segmented approximation problem can either be solved explicitly or at least asymptotically. So one may study the gain of accuracy and the optimal knot locations in detail.

EXAMPLE 1. Let $J = [1, 2]$, $f(x) = 1/x$ and $V_\nu = \Pi_1|_{J_\nu}$ for $\nu = 1, 2, \ldots, k$. For any numbers ξ, η with $1 \le \xi < \eta \le 2$ the best approximating polynomial p of f with respect to Π_1 on the interval $[\xi, \eta]$ is given by

$$p(x) = -\frac{x}{\xi\eta} + \frac{1}{2}\left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{\eta}}\right)^2.$$

Furthermore,

$$\rho(f; \xi, \eta) = \frac{1}{2}\left(\frac{1}{\sqrt{\xi}} - \frac{1}{\sqrt{\eta}}\right)^2.$$

In order to get a leveled partition of J, we have to satisfy the conditions

$$1 = \tau_0^* < \tau_1^* < \cdots < \tau_{k-1}^* < \tau_k^* = 2$$

and, with $z_\nu = 1/\sqrt{\tau_\nu^*}$,

$$z_{\nu+1} - 2z_\nu + z_{\nu-1} = 0,$$

for $\nu = 1, 2, \ldots, k-1$. Thus, we get the unique leveled partition T_k^* with the knots

$$\tau_\nu^* = \frac{1}{\left(1 - \left(2 - \sqrt{2}\right)\nu/2k\right)^2},$$

for $\nu = 0, 1, \ldots, k$. The knots are not equally distributed. There is a tendency to lower values.

The segmented minimal deviation is

$$\rho_k^*(f) = \frac{3 - 2\sqrt{2}}{4k^2}.$$

EXAMPLE 2. Let $J = [0,1]$, $f(x) = \sqrt{x}$ and $V_\nu = \Pi_1|_{J_\nu}$ for $\nu = 1, 2, \ldots, k$. The best approximating polynomial $q \in \Pi_1$ of f on the interval $[\xi, \eta]$, $0 \le \xi < \eta \le 1$ is given by

$$q(x) = \frac{x}{\sqrt{\xi} + \sqrt{\eta}} + \frac{\xi + 6\sqrt{\xi\eta} + \eta}{8\left(\sqrt{\xi} + \sqrt{\eta}\right)},$$

and furthermore,

$$\rho(f; \xi, \eta) = \frac{(\sqrt{\eta} - \sqrt{\xi})^2}{8(\sqrt{\xi} + \sqrt{\eta})}.$$

In order to get a leveled partition of the interval $[0,1]$, we have to satisfy the conditions

$$0 = \tau_0^* < \tau_1^* < \cdots < \tau_{k-1}^* < \tau_k^* = 1$$

and, with $z_\nu = \sqrt{\tau_\nu^*}$,

$$\frac{(z_\nu - z_{\nu-1})^2}{z_\nu + z_{\nu-1}} = \frac{(z_{\nu+1} - z_\nu)^2}{z_{\nu+1} + z_\nu},$$

for $\nu = 1, 2, \ldots, k-1$. A short computation yields

$$z_\nu = \frac{\nu(\nu+1)}{k(k+1)},$$

and hence, the unique leveled partition

$$\tau_\nu^* = \left(\frac{\nu(\nu+1)}{k(k+1)}\right)^2.$$

The knots are much closer to the left endpoint 0 of the interval J, a singular point of the function f. The segmented minimal deviation is

$$\rho_k^*(f) = \frac{1}{4k(k+1)}.$$

For this example, cp. [4, pp. 254–255; 5].

EXAMPLE 3. Let $\tilde{f}$ be a complex function of a complex variable z, holomorphic in some region $G \subset \mathbb{C}$, where G contains the real interval $J = [-1, +1]$. The restriction f of $\tilde{f}$ to the interval J is assumed to be real. An ellipse Γ with foci $+1$ and -1 is called the regularity ellipse of $\tilde{f}$, if $\tilde{f}$ is holomorphic in the interior of Γ and not in any larger ellipse with the same foci. Let us assume that $\tilde{f}$ possesses a finite regularity ellipse Γ and that the real positive point on Γ is σ. Of course we have $\sigma > 1$.

Let

$$\kappa = \kappa(f) = \sigma + \sqrt{\sigma^2 - 1}.$$

A famous result, due to Bernstein, states that the minimal deviation $E_n(f)$ in approximating the function f by polynomials of the space Π_n on the interval $[-1, +1]$ tends geometrically to zero:

$$\limsup_{n \to \infty} (E_n(f))^{1/n} = \frac{1}{\kappa}.$$

If we consider now any interval $[\xi, \eta]$ with $-1 \leq \xi < \eta \leq +1$ then, in order to compute the corresponding geometrical factor, one has to look for the largest ellipse $\bar{\Gamma}$ with foci ξ and η such that the function $\tilde{f}$ is holomorphic in its interior, and then to transform the interval by translation and scaling to the interval $[-1, +1]$. In what follows we restrict to a special case, in which we can easily compute the corresponding parameters explicitly. Let $\tilde{f}$ possess the two properties:

(a) there is a singularity at the point σ;
(b) the function $\tilde{f}$ is holomorphic in the disk

$$|z + 1| \leq 1 + \sigma.$$

Then the minimal deviation $\rho(f; \xi, \eta)$ in approximating the function f by polynomials of the space Π_n behaves geometrically with the parameter

$$\sigma(\xi, \eta) = \frac{2\sigma - \xi - \eta}{\eta - \xi}$$

instead of σ, and with the geometric factor

$$\kappa(\xi, \eta) = \sigma(\xi . \eta) + \sqrt{(\sigma(\xi, \eta))^2 - 1}$$

instead of κ. In order to get asymptotically for $n \to \infty$ a leveled partition, we choose such knots for which the subintervals J_ν all produce the same parameters; i.e., we have to solve the system of equations

$$\sigma\left(\tau_{\nu-1}^*, \tau_\nu^*\right) = \sigma\left(\tau_\nu^*, \tau_{\nu+1}^*\right),$$

for $\nu = 1, 2, \ldots, k - 1$. It is easy to verify that this system has the unique solution

$$\tau_\nu = \sigma - (\sigma + 1)\left(\frac{\sigma - 1}{\sigma + 1}\right)^{\nu/k},$$

for $\nu = 0, 1, \ldots, k$. The associated parameter and the geometric factor, respectively, for the asymptotically leveled partition are

$$\sigma_k^* = \frac{1 + ((\sigma - 1)/(\sigma + 1))^{1/k}}{1 - ((\sigma - 1)/(\sigma + 1))^{1/k}}$$

and

$$\kappa_k^* = \sigma_k^* + \sqrt{\sigma_k^{*2} - 1}.$$

EXAMPLE 4. Let $J = [a, b]$ with $0 < a < b$, $f(x) = \sqrt{x}$ and

$$V_\nu = \left\{\frac{1}{2}\left(p + \frac{x}{p}\right) \;\middle|\; p \in \Pi_n|_{J_\nu},\ p(x) > 0 \text{ for } x \in J_\nu\right\},$$

for $\nu = 1, 2, \dots, k$, where, in addition, we use the relative Chebyshev norm. For real numbers ξ, η with $a \le \xi < \eta \le b$ it is

$$\rho(f;\xi,\eta) = \inf_{p\in\Pi_n} \max_{x\in[\xi,\eta]} \left| \frac{\sqrt{x} - (1/2)\,(p(x) + x/p(x))}{\sqrt{x}} \right|.$$

We claim that $\rho(f;\xi,\eta)$ depends, for fixed n, only on the quotient ξ/η. In fact, for any $\lambda \in \mathbb{R}$, $\lambda > 0$, it follows easily that

$$\rho(f;\lambda\xi,\lambda\eta) = \rho(f;\xi,\eta).$$

If we intend to construct a leveled partition T_k^* with

$$a = \tau_0^* < \tau_1^* < \cdots < \tau_{k-1}^* < \tau_k^* = b,$$

we only have to look for those which satisfy

$$\frac{\tau_\nu^*}{\tau_{\nu-1}^*} = \frac{\tau_{\nu+1}^*}{\tau_\nu^*},$$

for $\nu = 1, 2, \dots, k-1$. This gives the knots

$$\tau_\nu^* = a \left(\frac{b}{a}\right)^{\nu/k}, \qquad \nu = 0, 1, \dots, k.$$

More details and generalizations to other best starting approximations can be found in [6]. We only present here the value of the segmented minimal deviation in the case $n = 1$. It is

$$\rho_k^*(f) = \frac{1 - \sqrt{1-\lambda_k^2}}{\sqrt{1-\lambda_k^2}},$$

where

$$\lambda_k = \left(\frac{(b/a)^{1/4k} - 1}{(b/a)^{1/4k} + 1}\right)^2.$$

A short computation yields the asymptotic relation

$$\rho_k^*(f) = \frac{\log^4(b/a)}{8192\,k^4} + O\left(k^{-5}\right), \qquad \text{for } k \to \infty.$$

4. SEGMENTED APPROXIMATION BY PÓLYA SPACES

For $n \in \mathbb{N}$, $\nu = 1, 2, \dots, n$, we consider real linear differential operators of order $\nu + 1$,

$$D_\nu u := \sum_{\mu=0}^{\nu+1} a_{\mu,\nu} u^{(\mu)},$$

where the following conditions are satisfied:

$$a_{\mu,\nu} \in C(J), \qquad \text{for } \nu = 0, 1, \dots, n; \quad \mu = 0, 1, \dots, \nu + 1,$$

and

$$a_{\nu+1,\nu} \ne 0, \qquad \text{for } x \in J.$$

Let $U_0, U_1, \dots, U_n$ be subspaces of $C(J)$ with

$$U_0 \subset U_1 \subset \cdots \subset U_n,$$

where U_ν coincides with the null space of the differential operator D_ν. We call U_n a *Pólya subspace* of $C(J)$ (cp. [7]). Obviously

$$\dim U_n = n + 1.$$

A theory of such subspaces can be found in [8]. There the equivalence, but for some differentiability properties, with those spaces, spanned by a so-called extended Chebyshev system (cf. [9,10]) has been discussed in detail. The case of constant coefficients $a_{\mu,\nu}$, in the context of segmented approximation, has been investigated by McLure [2].

Some properties of such Pólya subspaces, as far as they are of interest here, will be given in the following theorems. For the proofs of Theorems 6 and 7 we refer to the literature.

THEOREM 6. *[8,9] Let U_n be a Pólya subspace of $C(J)$. Then U_n is a Haar subspace on the interval J.*

We denote the minimal deviation in approximating a function $f \in C(J)$ with respect to the subspace U_n on the interval $]\xi, \eta]$ by $\rho_n(f; \xi, \eta)$.

THEOREM 7. *[3,8,11] Let U_n be a Pólya subspace of $C(J)$ with the corresponding differential operator D_n, and $[\xi, \eta]$ a subinterval of J. If for two functions $f, g \in C^{n+1}[\xi, \eta]$ the inequality*

$$|D_n f(x)| \leq D_n g(x) \tag{7}$$

holds for all $x \in [\xi, \eta]$, then we have

$$\rho_n(f; \xi, \eta) \leq \rho_n(g; \xi, \eta). \tag{8}$$

LEMMA 2. *[12] There exists a continuous function*

$$K : M \to \mathbb{R},$$

where

$$M := \left\{(x,t) \in \mathbb{R}^2 \mid a \leq t \leq x \leq b\right\},$$

possessing the following properties.

(1) *For every fixed $t \in [a, b]$, the function K is a solution of the homogeneous differential equation, with respect to x,*

$$D_n K = 0.$$

(2) *It is*

$$\left(\frac{\partial^\mu K}{\partial x^\mu}\right)_{x=t} = \begin{cases} 0, & \text{for } \mu = 0, 1, \ldots, n-1, \\ 1, & \text{for } \mu = n. \end{cases}$$

(3) *The function K is uniquely defined by the properties (1) and (2).*

(4) *It is*

$$K(x,t) > 0, \tag{9}$$

for all $(x,t) \in \mathbb{R}^2$ with $a \leq t < x \leq b$.

REMARK. The function $K(x,t)$ represents a generalization of the truncated power function, which occurs in the theory of polynomial splines, in the Peano kernel representation of linear functionals, etc.

PROOF OF LEMMA 2. For every fixed $t \in [a, b]$, the function K is the solution of a simple initial value problem with respect to the differential equation $D_n y = 0$. Hence, the function K is uniquely defined by (1) and (2) and depends continuously on the parameter t. Obviously it is also continuous on M.

Now let $t \in [a, b)$ be fixed. Since the null space U_n of the differential equation $D_n y = 0$ is a Haar space of dimension $n + 1$ with

$$U_n \subset C^{n+1}(J),$$

every function $u \in U_n$ possesses at most n zeros in $[t, b)$ or vanishes identically. Due to differentiability properties of u, the zeros may be counted according to their multiplicity, but at most n times. In our case, the function K, considered as a member of U_n, has all its n zeros at the point $x = t$. Since the n^{th} partial derivative of K at the point $x = t$ equals 1, the function K is positive for all $(x, t) \in \mathbb{R}^2$ with $a \le t < x \le b$. ∎

THEOREM 8. *Let U_n be a Pólya subspace of $C(J)$ and $[\xi, \eta]$ be a subinterval of J. Furthermore let $f \in C^{n+1}(J)$. Then there is a value ζ with $\xi \le \zeta \le \eta$ such that*

$$\rho_n(f; \xi, \eta) = \rho_n(f_0; \xi, \eta) \, |D_n f(\zeta)| \,. \tag{10}$$

Here f_0 is any solution of the differential equation

$$D_n y = 1$$

on J. One possible choice for f_0 is given by

$$f_0(x) = \int_\xi^x K(x, t) \, dt,$$

where K denotes the function in Lemma 2.

PROOF. If $D_n f(x) > 0$ for all $x \in [\xi, \eta]$, then, according to Theorem 7,

$$\rho_n(f_0; \xi, \eta) \min_{x \in [\xi, \eta]} D_n f(x) \le \rho_n(f; \xi, \eta) \le \rho_n(f_0; \xi, \eta) \max_{x \in [\xi, \eta]} D_n f(x).$$

Then there is a number γ with

$$\min_{x \in [\xi, \eta]} D_n f(x) \le \gamma \le \max_{x \in [\xi, \eta]} D_n f(x)$$

such that

$$\rho_n(f; \xi, \eta) = \rho_n(f_0; \xi, \eta) \, \gamma.$$

Due to the continuity of $D_n f(x)$ there exists a number ζ with $\xi \le \zeta \le \eta$ so that

$$\gamma = D_n f(\zeta).$$

If $D_n f(x) < 0$ for all $x \in [\xi, \eta]$, we just replace f by $-f$ to get the same result. If

$$0 \in \left[\min_{x \in [\xi, \eta]} D_n f(x), \max_{x \in [\xi, \eta]} D_n f(x) \right],$$

then it follows, using Theorem 7 again, that there is a number κ with

$$\kappa \le \max \left\{ \left| \min_{x \in [\xi, \eta]} D_n f(x) \right|, \max_{x \in [\xi, \eta]} D_n f(x) \right\}$$

such that

$$\rho_n(f; \xi, \eta) = \rho_n(f_0; \xi, \eta) \, \kappa$$

holds. Due to the continuity of $D_n f(x)$, there is a number ζ with $\xi \le \zeta \le \eta$ such that

$$\kappa = |D_n f(\zeta)| \,.$$

Obviously the function

$$\int_{\xi}^{x} K(x,t)\,dt$$

satisfies the differential equation $D_n y = 1$. ∎

THEOREM 9. *Let U_n be a Pólya subspace of $C(J)$. Furthermore let $f \in C^{n+1}(J)$. Then there is a number $\zeta^* \in J$ such that the segmented minimal deviation satisfies*

$$\rho_{n.k}^*(f) \le \frac{L_n (b-a)^{n+1}}{(n+1)!\,k^{n+1}} \left| D_n f(\zeta^*) \right|, \tag{11}$$

where

$$L_n = \sup_{(x,t)\in M} \left| \frac{\partial^n K}{\partial x^n} \right|$$

and K denotes the function from Lemma 2.

PROOF. Let $n \in \mathbb{N}$. The function K from Lemma 2 possesses the properties

$$\left(\frac{\partial^\mu K}{\partial x^\mu} \right)_{x=t} = 0, \qquad \text{for } \mu = 0, 1, \ldots, n-1.$$

Hence, Taylor's Theorem yields the formula

$$K(x,t) = \frac{1}{(n-1)!} \int_t^x \frac{\partial^n K(u,t)}{\partial u^n} (x-u)^{n-1}\,du.$$

Thus,

$$0 \le K(x,t) \le L_n \frac{(x-t)^n}{n!}, \qquad \text{for } (x,t) \in M, \tag{12}$$

with

$$L_n = \sup_{(x,t)\in M} \left| \frac{\partial^n K}{\partial x^n} \right|.$$

It follows that

$$\int_{\xi}^{\eta} K(x,t)\,dt \le L_n \frac{(\eta-\xi)^{n+1}}{(n+1)!},$$

and therefore, with

$$f_0(x) = \int_{\xi}^{x} K(x,t)\,dt,$$

the inequality

$$\rho_n(f_0; \xi, \eta) \le \|f_0\|_{[\xi,\eta]} \le L_n \frac{(\eta-\xi)^{n+1}}{(n+1)!}. \tag{13}$$

Now let T_k^* be a leveled partition of J. The formula (10) gives the existence of numbers $\zeta_\nu \in J_\nu$, $\nu = 1, 2, \ldots, k$ such that

$$\rho_{n,k}^*(f) = \rho_n\left(f; \tau_{\nu-1}^*, \tau_\nu^*\right) = \rho_n\left(f_0; \tau_{\nu-1}^*, \tau_\nu^*\right) \left| D_n f(\zeta_\nu) \right|,$$

for $\nu = 1, 2, \ldots, k$. Hence,

$$\left(\rho_{n,k}^*(f)\right)^{1/(n+1)} = \frac{1}{k} \sum_{\nu=1}^{k} \left| D_n f(\zeta_\nu) \right|^{1/(n+1)} \left(\rho_n\left(f_0; \tau_{\nu-1}^*, \tau_\nu^*\right) \right)^{1/(n+1)}.$$

Now, with

$$f_0(x) = \int_{\tau_{\nu-1}^*}^{x} K(x,t)\,dt,$$

we get from (13) the estimation

$$\rho_n\left(f_0;\tau_{\nu-1}^*,\tau_\nu^*\right) \le \|f_0\| \le \frac{L_n}{(n+1)!}\left(\tau_\nu^* - \tau_{\nu-1}^*\right)^{n+1}.$$

Therefore,

$$\left(\rho_{n,k}^*(f)\right)^{1/(n+1)} \le \left(\frac{L_n}{(n+1)!}\right)^{1/(n+1)} \frac{(b-a)}{k} \sum_{\nu=1}^{k} |D_n f(\zeta_\nu)|^{1/(n+1)} \left(\frac{\tau_\nu^* - \tau_{\nu-1}^*}{b-a}\right).$$

The last sum represents a convex combination of function values of

$$|D_n f(x)|^{1/(n+1)}.$$

Hence, there exists, due to continuity, a number $\zeta^* \in J$ such that

$$\sum_{\nu=1}^{k} |D_n f(\zeta_\nu)|^{1/(n+1)} \left(\frac{\tau_\nu^* - \tau_{\nu-1}^*}{b-a}\right) = |D_n f(\zeta^*)|^{1/(n+1)}. \tag{14}$$

Now the assertion of Theorem 9 follows easily. ∎

LEMMA 3. *Let U_n be a Pólya subspace of $C(J)$. The function $f \in C(J)$ is assumed to possess the property: There is no interval $[\alpha,\beta] \subset J$ with $\alpha < \beta$ such that $f|_{[\alpha,\beta]}$ is an element of U_n on this interval. For $k \in \mathbb{N}$, $k \ge 2$ let*

$$T_k^* = \left\{\tau_{1,k}^*, \tau_{2,k}^*, \dots, \tau_{k-1,k}^*\right\}$$

be a sequence of leveled partitions of J with respect to f and

$$\omega_k = \max_{\nu=1,\dots,k-1}\left(\tau_{\nu,k}^* - \tau_{\nu-1,k}^*\right).$$

Then

$$\lim_{k\to\infty} \omega_k = 0.$$

PROOF. Let $\gamma > 0$ be a limit point of a subsequence of the sequence $\{\omega_k\}_{k\in\mathbb{N}}$. Then there is a further subsequence J_{ν_k} of subintervals and an interval $[\alpha,\beta]$, $\alpha < \beta$, such that

$$[\alpha,\beta] \subset J_{\nu_k}$$

holds for all sufficiently large values of k. From the inequality (11), we know that

$$\lim_{k\to\infty} \rho_{n,k}(f) = 0.$$

Hence, $f|_{[\alpha,\beta]} \in U_n$. But this possibility was excluded. Therefore, the sequence $\{\omega_k\}_{k\in\mathbb{N}}$ does not possess a positive point of accumulation γ. ∎

THEOREM 10. *Under the assumption of Lemma 3 we have*

$$\rho_{n,k}^*(f) \le \frac{L_n}{(n+1)!k^{n+1}}\left(\int_a^b |D_n f(t)|^{1/(n+1)}\,dt\right)^{n+1}(1+o(1)), \tag{15}$$

for $k \to \infty$. Here L_n is the constant in Theorem 9.

PROOF. We conclude from the proof of Theorem 9 that

$$\rho_{n,k}^{*}(f)^{1/(n+1)} \leq \left(\frac{L_n}{(n+1)!}\right)^{1/(n+1)} \frac{1}{k} \Lambda_{n,k}(f) \tag{16}$$

holds with

$$\Lambda_{n,k}(f) = \sum_{\nu=1}^{k} |D_n f(\zeta_{\nu,k})|^{1/(n+1)} \left(\tau_{\nu,k}^{*} - \tau_{\nu-1,k}^{*}\right),$$

with values $\zeta_{\nu,k} \in J_\nu$, $\nu = 1, 2, \ldots, k$. According to Lemma 3, the quantity $\Lambda_{n.k}(f)$ is a Riemann sum converging to the integral

$$\lim_{k\to\infty} \Lambda_{n,k}(f) = \int_a^b |D_n f(t)|^{1/(n+1)}\, dt. \tag{17}$$

The formulas (16) and (17) yield the assertion. ∎

REMARK 1. It is easy to prove that, for every $k \in \mathbb{N}$, there is only one leveled partition, if

$$D_n f(x) \neq 0$$

holds for all $x \in J$.

REMARK 2. It is obvious that better information on the minimal deviation

$$\rho_n(f_0; \xi, \eta)$$

will produce better estimates of the segmented minimal deviation $\rho_{n,k}^{*}(f)$. Even lower bounds can be derived, as we will see in the next section.

5. SEGMENTED APPROXIMATION BY WEIGHTED POLYNOMIAL SPACES

Let $w \in C^{n+1}(J)$ be a fixed real function, which is positive on J. We consider the spaces

$$\Pi_n(w) := \{w\, p \mid p \in \Pi_n\},$$

where Π_n denotes, as usual, the space of polynomials of degree at most n. Obviously, $\Pi_n(w)$ is a Pólya subspace of $C(J)$. These spaces are important and very convenient in many concrete applications. The differential operator, belonging to the space $\Pi_n(w)$, is given by

$$D_n y = \frac{d^{n+1}}{dx^{n+1}} \left(\frac{y}{w}\right). \tag{18}$$

For the auxiliary function f_0 in Theorem 8, we may choose

$$f_0(x) = \frac{x^{n+1} w(x)}{(n+1)!}.$$

Hence,

$$\rho_n(f_0; \xi, \eta) = \rho(f_0; \xi, \eta; w) = \frac{1}{(n+1)!} \inf_{p\in\Pi_n} \left\| w\left(x^{n+1} - p\right)\right\|_{[\xi,\eta]}.$$

We therefore are looking for the minimal deviation of weighted Chebyshev polynomials on the interval $[\xi, \eta]$.

The simplest case occurs for $w \equiv 1$. Here, using the well-known properties of the classical Chebyshev polynomials, one gets

$$\rho_n(f_0; \xi, \eta; 1) = \frac{(\eta - \xi)^{n+1}}{2^{2n+1}(n+1)!}.$$

A trivial result is

$$\rho_n(f_0;\xi,\eta;w) = c_n(w)\frac{(\eta-\xi)^{n+1}}{2^{2n+1}(n+1)!}$$

with

$$\min_{x\in J} w(x) \le c_n(w) \le \max_{x\in J} w(x).$$

The quantity $c_n(w)$ depends on ξ and η. It can be proved (cp. [13]) that

$$\lim_{n\to\infty} c_n(w) = e^{\frac{1}{\pi}\int_\xi^\eta \left(\log w(x)/\left(\sqrt{(t-\xi)(\eta-t)}\right)\right)\,dt}$$

holds true. If $1/w$ is the square root of a positive polynomial $q \in \Pi_n$, the quantity $c_n(w)$ can be computed explicitly (cp. [13]). In any case, a careful analysis gives narrow bounds for $c_n(w)$ (cp. [8, Problem 4.6]).

THEOREM 11. *Let $f \in C^{n+1}(J)$ and $V = \Pi_n(w)$. Then there exist two numbers $\xi^*, \zeta^* \in J$ such that*

$$\rho^*_{n,k}(f) = \frac{\left|(f/w)^{(n+1)}(\zeta^*)\right| w(\xi^*)}{2^{2n+1}(n+1)!\,k^{n+1}}(b-a)^{n+1}. \tag{19}$$

REMARK. For $w \equiv 1$ we get the classical case of polynomial segmental approximation. The formula (19) has been proved in this case by Phillips [14] in 1970. It was the first result in this direction. The formula (19) could be derived directly from Phillips' result also.

PROOF OF THEOREM 11. As in the proof of Theorem 9 we have

$$\left(\rho^*_{n,k}(f)\right)^{1/(n+1)} = \frac{1}{k}H_{n,k}(f),$$

where

$$H_{n,k}(f) = \sum_{\nu=1}^{k}\left|\left(\frac{f}{w}\right)^{(n+1)}(\zeta_\nu)\right|^{1/(n+1)} \rho_n\left(f;\tau^*_{\nu-1},\tau^*_\nu;w\right)^{1/(n+1)}.$$

Hence

$$\left(\frac{\min_{x\in J} w(x)}{2^{2n+1}(n+1)!}\right)^{1/(n+1)} S_{n,k}(f) \le H_{n,k}(f) \le \left(\frac{\max_{x\in J} w(x)}{2^{2n+1}(n+1)!}\right)^{1/(n+1)} S_{n,k}(f).$$

Here

$$S_{n,k}(f) = \sum_{\nu=1}^{k}\left|\left(\frac{f}{w}\right)^{(n+1)}(\zeta_\nu)\right|^{1/(n+1)}\left(\tau^*_\nu - \tau^*_{\nu-1}\right) = (b-a)\left|\left(\frac{f}{w}\right)^{(n+1)}(\zeta^*)\right|^{1/(n+1)}, \tag{20}$$

for some $\zeta^* \in J$. Thus, there is a number $\xi^* \in J$ such that

$$\left(H_{n,k}(f)\right)^{n+1} = \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!}\left|\left(\frac{f}{w}\right)^{(n+1)}(\zeta^*)\right| w(\xi^*).$$

This proves the assertion (19). ∎

THEOREM 12. *Under the assumptions of Lemma 3, where the space U_n has to be replaced by the space $\Pi_n(w)$, there exists a number $\eta^* \in J$ such that*

$$\rho^*_{n,k}(f) = \frac{w(\eta^*)}{2^{2n+1}(n+1)!\,k^{n+1}}\left(\int_a^b \left|\left(\frac{f}{w}\right)^{(n+1)}(t)\right|^{1/(n+1)} dt\right)^{n+1}(1+o(1)), \tag{21}$$

for $k\to\infty$.

REMARK. Formula (21) has been proved in the case $w \equiv 1$ by Phillips [14] in 1970 (cp. also [15–17]). The formula (21) could be derived directly from Phillips' result also. An analogue of (21) in the case that the operator D_n, defined in Section 4, has constant coefficients and only real zeros of its characteristic polynomial, is due to McLure [2].

PROOF OF THEOREM 12. The sum $S_{n,k}(f)$ in (20), formed for a sequence T_k^* of leveled partitions, is a Riemann sum which, under our assumptions, converges to the integral

$$\int_a^b \left| \left(\frac{f}{w}\right)^{(n+1)} (t) \right|^{1/(n+1)} dt.$$

Now the assertion (21) follows easily. ∎

If we are interested to use segmental approximations by weighted polynomials on unbounded intervals, Theorems 11 and 12 are worthless. One example is as follows: take $J = [0, \infty)$, $V = \Pi_n(w)$ with $w(x) = e^{-x}$. This is a reasonable approximation problem. One main difficulty here is that nothing is known about the minimal deviation

$$\rho_n \left(f; \xi, \eta; e^{-x}\right).$$

There is only a conjecture in the special case $\xi = 0$, $\eta = \infty$, (cp. [18]).

6. SOME REMARKS ON CONSTRUCTIVE METHODS

Until now, every known method of construction for optimal segmented approximations consists of constructing a leveled partition of the corresponding interval. The first approaches, in the polynomial case, are based on the following idea: if f satisfies the assumptions of Lemma 3 with $U_n = \Pi_n$, and if k is large, then one may try to solve the system of equations

$$\left| f^{(n+1)} \left(\frac{\tau_{\nu-1} + \tau_\nu}{2}\right) \right| = \left| f^{(n+1)} \left(\frac{\tau_\nu + \tau_{\nu+1}}{2}\right) \right|,$$

$\nu = 1, 2, \ldots, k-1$ (cp. [15,16,19–23]). If this method can be applied, an *a posteriori* estimation of $\rho_{n,k}^*(f)$, according to Theorem 3, helps to justify it numerically. If f is the restriction of an analytic function (cp. Example 3), special knowledge of the location and the kind of singularities of the analytic function may be quite useful and time saving in getting information on a leveled partition of the interval J. Such considerations can even modify the approximating spaces, in as giving ideas for better weight functions.

In 1986, Nürnberger, Sommer and Strauß [24] developed an iterative algorithm which converges, for all kind of subsets V_ν and for every $f \in C(J)$, to an optimal segmented approximation on J. Furthermore, the use of certain linear functionals which are good substitutes for the minimal deviations $\rho_n(f; \xi, \eta)$, provided $V_\nu = \Pi_n|_{[\xi,\eta]}$, can simplify the procedure considerably. This has been investigated, with emphasis on the polynomial case, by Meinardus, Nürnberger, Sommer and Strauß [25]. This simplification of the algorithm can possibly be generalized to Pólya subspaces: one has to construct first the best approximation to the function

$$f_0(x) = \int_\xi^x K(x,t)\, dt$$

with respect to the subspace $U_n|_{[\xi,\eta]}$ on the interval $[\xi, \eta]$, if possible, and then compute a linear functional based on the unique alterant of f_0, which vanishes on the subset $U_n|_{[\xi,\eta]}$ and has functional norm equal to 1. This will give, in general, a good substitute for the minimal deviation in question and is much easier to evaluate.

Leveled partitions of J with $V = \Pi_n$ may serve as a good choice in constructing best spline approximations with free knots to a given function. This has been pointed out by Nürnberger [26] and in [25].

There are several interesting problems in optimal partitioning sets in $\mathbb{R}^d$ with respect to the approximation of given multivariate functions; cp. [5,27].

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